Sonic Boom Effects on Beams Loosely Bound to Their Supports

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In an effort to explain potential damage to structural elements caused by sonic booms, the effect of a gap between the element and its supports is considered. This is studied in detail in the response of a beam allowed to rattle between two sets of springs and subjected to an N-shaped pressure pulse. Damping is neglected. The response is expressed in either one of two normal function series, depending on whether the beam is in contact with its supports or not. For moderately stiff springs, the dynamic amplification factors are considerably higher than for a firmly supported beam; the dynamic amplification factor for shear increases very rapidly as the relative spring stiffness increases. Other factors considered are ratio of sonic boom duration to fundamental period of beam, ratio of gap to beam span, and ratio of static deflection to beam span. This last ratio affects the dynamic amplification factor, thus showing that the maximum response is not proportional to the peak pressure. The problem is therefore nonlinear.

Nomenclature

A	$= a/L$
a	$= \frac{1}{2}$ width of gap
$D_n{}^f(t)$, $D_n{}^s(t)$	= time-dependent factors in the response
E	= beam modulus of elasticity
f, s	= superscripts denoting free-free or spring-supported phase
$H(t)$	= unit step function $= 0$ if $t < 0$; $= 1$ otherwise
I	= moment of inertia of beam
K	$= EI/SL^3$
L	= beam span
M	= bending moment
m, n	= subscripts
$P(t)$	= forcing function
$p_n{}^f$, $p_n{}^s$	= circular frequencies
p_0	= maximum overpressure in N-wave
q	= phase indicator
R	$= \tau/T_f$
R_d	$= p_0L^3/EI$
S	= support spring const
t	= time
T_f	= fundamental period of free-free beam
T_{nm}	$= \int_{-L/2}^{L/2} Z_n{}^s(x)Z_m{}^f(x)dx$
v	= velocity
Q	= shear force
x	= abscissa along beam axis measured from the origin (at midspan)
$Z^f(x)$, $Z^s(x)$	= normal functions of free-free and spring-supported beam, respectively
$Z(x,t)$	= beam deflection
$\|Z\|$	$= \left(\int_{-L/2}^{L/2} [Z(x)]^2dx\right)^{1/2}$
$\beta_n{}^s,\beta_n{}^f$	= frequency coefficients
μ	= mass/unit length
τ	= sonic boom duration
δ_{mn}	= Kronecker delta

Introduction

In houses of substandard construction or in disrepair, structural elements such as window panes are often found not to be in immediate contact with the supporting frame. A disturbance in the form of a pressure pulse will induce rattling of the element within the gap, possibly resulting in severe dynamic effects. Although ample evidence of wind damage exists, no treatment of such problems has been found in the literature. A detailed treatment of the rattling of a beam is presented in this paper.

The pressure pulse considered will be an N-shaped pressure pulse corresponding to a typical far-field sonic boom disturbance (Fig. 1). The response of a simply supported beam to such a disturbance has been studied in detail.[1] In the present paper, the same N-pressure pulse is considered to act on a beam whose end supports are pairs of springs having a gap within which the beam may freely move (Fig. 2). A comparison of these responses allows one to ascertain the effect of rattling on the dynamic amplification factors. It is believed that the solution of the rattling beam problem provides some insight into the problem of a rattling plate.

There is an obvious advantage in studying the behavior of a simple beam on spring supports. The spring supports, simulating the elastic behavior of the supporting frame, can be as flexible or as stiff as the case may be, with the exception of rigid or nearly rigid supports for which the Bernoulli-Euler theory that is used throughout becomes unrealistic.[2] Damping is not considered.

For a given beam, the dynamic amplification factors will depend on the spring stiffness, the sonic boom duration, the gap at the supports, and the maximum overpressure. The effect of each of these quantities (expressed in a dimensionless ratio) is studied by subjecting each to a continuous or discrete variation while keeping the others constant. The results are presented in the forms of graphs and tables.

Formulation of the Problem

With reference to the Nomenclature, the equation of motion, according to the Bernoulli-Euler theory of flexure without damping, is[3]

$$EIZ'''' + \mu\ddot{Z} = P(t) \tag{1}$$

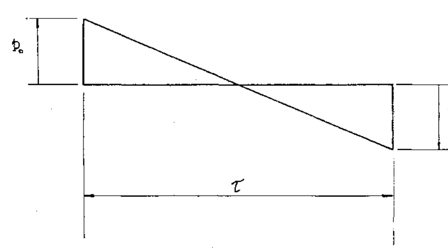

Fig. 1. Typical sonic boom disturbance.

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Let q be an integer whose initial value is 0 and which increases by one every time the beam either comes into contact or loses contact with the springs. It is then obvious that, depending on whether this phase indicator q is even or odd, the beam is in a free-free or spring-supported phase of its rattling motion.

The solution of (1) will be taken in the form

$$Z(x,t) = \sum_{n=1}^{\infty} Z_n{}^f(z) D_n{}^f(t) \qquad \text{if } q = \text{even} \tag{2a}$$

$$Z(x,t) = \pm a + \sum_{n=1}^{\infty} Z_n{}^s(x) D_n{}^s(t) \qquad \text{if } q = \text{odd} \tag{2b}$$

where the sign in (2b) depends on whether the beam is in contact with the lower or upper springs. The functions $Z_n{}^f(x)$ and $Z_n{}^s(x)$ are the normal functions of the free-free and spring-supported beam, respectively. These functions are determined together with the corresponding frequencies $p_n{}^f$ and $p_n{}^s$ in Appendix A.

The orthogonality properties of the normal functions allow the determination of the time factors $D_n(t)$ through the ordinary differential equation,

$$\ddot{D}_n + p_n{}^2 D_n = c_n[P(t)/\mu] \tag{3}$$

where c_n is the Fourier coefficient of the function equal to 1 in the interval $(-L/2, L/2)$, and which is computed in Appendix B. The superscripts f and s have been deleted in (3) since the same equation applies in both cases.

The initial conditions to be used with (3) are determined inductively. Let $t = t_q$ be the beginning of phase q. Then obviously $t_0 = 0$, and $D(t_0) = \dot{D}(t_0) = 0$. Suppose that the response in phase $q - 1$ has been determined; then t_q is obtained as the smallest root exceeding t_{q-1} of $|Z(\pm L/2,t)| = a$. For $t = t_q$, the displacement and velocity of every point of the beam must be continuous. Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} Z_n{}^f(x) D_n{}^f(t_q) &= \sum_{n=1}^{\infty} Z_n{}^s(x) D_n{}^s(t_q) \pm a \\ \sum_{n=1}^{\infty} Z_n{}^f(x) \dot{D}_n{}^f(t_q) &= \sum_{n=1}^{\infty} Z_n{}^s(x) \dot{D}_n{}^s(t_q) \end{aligned} \tag{4}$$

Making use of the orthogonality properties of the normal functions, the following relations are obtained:

$$\begin{aligned} D_n{}^s(t_q) &= \frac{1}{\|Z_n{}^s\|^2} \sum_{m=1}^{\infty} T_{nm} \left(D_m{}^f(t_q) \mp \delta_{1m} \frac{a}{2} \right) \\ \dot{D}_n{}^s(t_q) &= \frac{1}{\|Z_n{}^s\|^2} \sum_{m=1}^{\infty} T_{nm} \dot{D}_m{}^f(t_q) \\ &(q = \text{odd}) \end{aligned} \tag{5}$$

$$\begin{aligned} D_n{}^f(t_q) &= \frac{1}{\|Z_n{}^f\|^2} \sum_{m=1}^{\infty} T_{mn} D_m{}^s(t_q) \pm \delta_{1n} \frac{a}{2} \\ \dot{D}_n{}^f(t_q) &= \frac{1}{\|Z_n{}^f\|^2} \sum_{m=1}^{\infty} T_{mn} \dot{D}_m{}^s(t_q) \\ &(q = \text{even}) \end{aligned} \tag{6}$$

where

$$\|Z_n{}^s\|^2 = \int_{-L/2}^{L/2} [Z_n{}^s(x)]^2 dx \tag{7}$$

$$\|Z_n{}^f\|^2 = \int_{-L/2}^{L/2} [Z_n{}^f(x)]^2 dx \tag{8}$$

$$T_{nm} = \int_{-L/2}^{L/2} Z_n{}^s(x) Z_m{}^f(x) dx \tag{9}$$

Equations (5) and (6) allow the determination of the initial values $D(t_q)$ and $\dot{D}(t_q)$, since the time factors D occurring on the right side correspond to phase $q - 1$ and are therefore known.

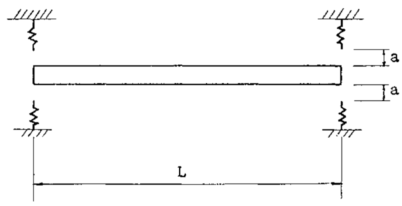

Fig. 2 **Initial state of beam.**

With the initial values of $D(t_q)$ and $\dot{D}(t_q)$ thus determined, the solution of (3) for $t_q \leq t \leq t_{q+1}$ may be obtained by Duhamel's integral,[3]

$$D_n(t) = \frac{\dot{D}_n(t_q)}{p_n} \sin p_n(t - t_q) + D_n(t_q) \cos p_n(t - t_q) + \frac{c_n}{\mu p_n} \int_{t_q}^{t} \sin p_n(t - t') P(t') dt' \tag{10}$$

For the particular case of $p_n = 0$, $\sin p_n(t - t_q)/p_n$ should be replaced by $(t - t_q)$.

The evaluation of $\dot{D}_n(t)$ is obtained by differentiation,

$$\dot{D}_n(t) = \dot{D}_n(t_q) \cos p_n(t - t_q) - D_n(t_q) p_n \sin p_n(t - t_q) + \frac{c_n}{\mu} \int_{t_q}^{t} \cos p_n(t - t') P(t') dt' \tag{11}$$

Response of the Beam to an N-Shaped Pulse

Let the N-shaped pressure pulse be represented by

$$P(t') = p_0(1 - 2t'/\tau) H(\tau - t') \tag{12}$$

where $H(\tau - t')$ is the Heaviside step function having the value of unity when $t' \leq \tau$, and zero when $t' > \tau$. Substituting (12) into (10) and (11), and carrying out the integration, it is found that for a spring-supported phase,

$$\begin{aligned} D_n(t) = {} & \frac{\dot{D}_n(t_q)}{p_n} \sin p_n(t - t_q) + D_n(t_q) \cos p_n(t - t_q) + \\ & \frac{c_n p_0}{\mu} H(\tau - t_q) \left\{ \left[\frac{\cos p_n(t - \tau)}{p_n{}^2} + \frac{2 \sin p_n(t - \tau)}{p_n{}^3 \tau} \right] \times \right. \\ & [H(\tau - t) - 1] - \frac{\cos p_n(t - t_q)}{p_n{}^2} \left(1 - \frac{2t_q}{\tau}\right) + \\ & \left. \frac{2 \sin p_n(t - t_q)}{p_n{}^3 \tau} + \frac{1}{p_n{}^2} \left(1 - \frac{2t}{\tau}\right) H(\tau - t) \right\} \end{aligned} \tag{13}$$

and

$$\begin{aligned} \dot{D}_n(t) = {} & \dot{D}_n(t_q) \cos p_n(t - t_q) - p_n D_n(t_q) \sin p_n(t - t_q) + \\ & \frac{c_n p_0}{\mu} H(\tau - t_q) \left\{ \left[-\frac{\sin p_n(t - \tau)}{p_n} + \frac{2 \cos p_n(t - \tau)}{p_n{}^2 \tau} \right] \times \right. \\ & [H(\tau - t) - 1] + \frac{\sin p_n(t - t_q)}{p_n} \left(1 - \frac{2t_q}{\tau}\right) + \\ & \left. \frac{2 \cos p_n(t - t_q)}{p_n{}^2 \tau} - \frac{2}{p_n{}^2 \tau} H(\tau - t) \right\} \end{aligned} \tag{14}$$

For a free-free phase, the previous formulas hold, with $c_n = 0$ for $n > 1$. For $n = 1$, $p_1 = 0$ and $c_1 = \frac{1}{2}$ (see Appendix B). The response is found by taking the limits of (13) and (14) as p_1 goes to 0. Thus

$$\begin{aligned} D_1(t) = {} & \dot{D}_1(t_q)(t - t_q) + D_1(t_q) + \frac{p_0}{2\mu} h(\tau - t_q) \times \\ & \left\{ \left[-\frac{(t - \tau)^2}{2} - \frac{(t - \tau)^3}{3\tau} \right] [H(\tau - t) - 1] + \right. \\ & \left. \frac{(t - t_q)^2}{2} \left(1 - \frac{2t_q}{\tau}\right) - \frac{(t - t_q)^3}{3\tau} \right\} \end{aligned} \tag{15}$$

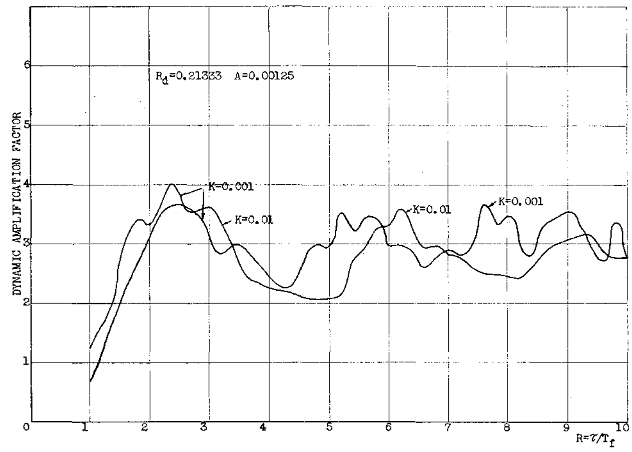

Fig. 3 DAF for moment.

and

$$\dot{D}_1(t) = \dot{D}_1(t_q) + \frac{p_0}{2\mu} H(\tau - t_q)\left\{[-(t-\tau) - (t-\tau)^2] \times [H(\tau - t) - 1] + (t - t_q)\left(1 - \frac{2t_q}{\tau}\right) - \frac{(t-t_q)^2}{3\tau}\right\} \quad (16)$$

With the values given by (13) and (15), Eq. (2) can be used to compute the values of end deflection and shear, as well as those of the center deflection and moment. The computations have been carried out by using 5 terms of each infinite series at time intervals of $i\tau/20$ where i is an integer. Because of the continuous rattling of the beam subsequent to the passing of the disturbance, all computations have been carried out up to the time $\tau + 3T_f$. The maximum dynamic amplification factors thus found for the center moment and the end shear for several combinations of pertinent parameters are presented in Figs. 3 and 4, and in Tables 1–3.

Choice of Parameters

The necessary parameters for the determination of the amplification factors are: 1) the ratio of stiffness of beam and spring support, $K = EI/SL^3$; 2) the ratio of sonic boom duration to fundamental period of a free-free beam, $R = \tau/T_f$; 3) the ratio of half-gap to beam span, $A = a/L$; and 4) $R_d' = 5R_d/384$ with $R_d = p_0L^3/EI$.

It would be desirable to vary each of the parameters continuously within its range of interest, in such a manner as to ascertain the absolute maximum dynamic response of the structural element. However, it is clear that the computation effort required would be extremely large, though the computations present no difficulty. It is considered reasonable to make a certain discrete variation for some of the parameters and continuous variation for others so that at least a qualitative understanding may be gained from the results.

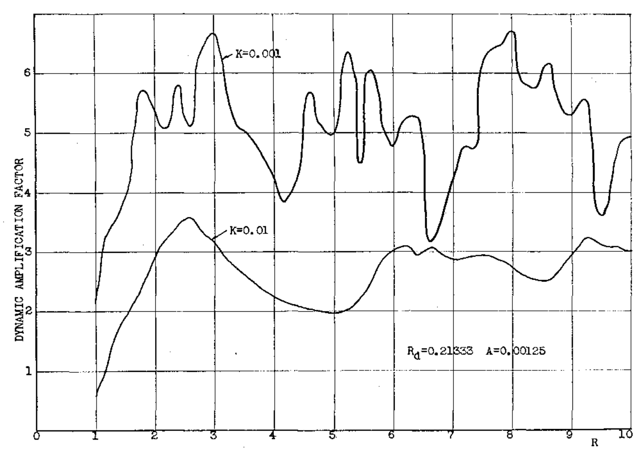

Fig. 4 DAF for shear.

Table 1 DAF vs K

$R_d = 0.42667$	$A = 0.00125$ Center deflection	Shear	$R = 2.0$ Moment
$K = 0.0001$	2.123	9.498	3.549
$K = 0.001$	2.181	3.580	3.048
$K = 0.01$	1.978	2.739	3.083

Three values of K were selected in the numerical computations, $K = 0.01$, 0.001, and 0.0001. Since the center deflection of the beam under a uniform load w is $5wL^4/384\ EI$ and the deflection of the spring support is $wL/2S$, the K values so selected correspond to the ratio of these deflections equal to 2.6, 26, and 260, respectively. But for $K = 0.0001$, it was discovered that, for moderate values of R, A, and R_d, the beam rattles so fast that a drastic decrease of time interval must be effected in order to describe adequately its behavior. Also, the beam spring system acquires frequencies comparable to a beam hitting rigid supports; hence, it became doubtful that a five-term series that was used throughout would adequately account for the correct behavior of the beam. In addition, there is a very real possibility that the Bernoulli-Euler theory would be inadequate for such stiff springs.

The choice of R as the ratio of sonic boom duration to fundamental period of a free-free beam instead of a simple beam is arbitrary but convenient, because the fundamental frequency of a free-free beam arises naturally in this problem. Besides, the fundamental period of a simple beam is very nearly twice that of a free-free beam; hence, the conversion would be rather simple. The range of R has been chosen to be from 1 to 10. It corresponds to a range of 0.5–5 if the fundamental period of a simple beam instead of a free-free beam is used for the ratio.

The beam has been assumed to be at the center of the gap at time zero, when the disturbance begins. Three values of gap representing 25, 50, and 100% of the thickness of a realistic window pane were selected. They correspond, respectively, to 0.0003215, 0.000625, and 0.00125 for the ratio A.

In selecting the ratio of maximum static deflection to span of beam, it was recognized that the window pane is not permitted to be designed to have this ratio exceed $\frac{1}{175}$ for wind load.[4] Since the sonic boom pressure is expected to be much less than the wind pressure, two values of the ratios, i.e., $\frac{1}{360}$ and $\frac{1}{180}$ were selected. They correspond to R_d equal to 0.21333 and 0.42667, respectively.

Summary and Conclusion

The problem of a beam induced to rattle between two sets of springs by sonic boom disturbances is formulated and a

Table 2 DAF vs R_d

$K = 0.01$	$A = 0.00125$ Center deflection	$R = 2.0$ Shear	Moment
$R_d = 0.21333$	1.792	2.930	3.079
$R_d = 0.42667$	1.968	2.739	3.083
$K = 0.01$	$A = 0.00125$ Center deflection	$R = 3.0$ Shear	Moment
$R_d = 0.21333$	2.088	3.166	3.607
$R_d = 0.42667$	1.966	2.439	3.094

Table 3 DAF vs A

$R_d = 0.42667$	$R = 2.0$ Center deflection	$K = 0.01$ Shear	Moment
$A = 0.0003215$	2.051	2.131	2.616
$A = 0.000625$	2.019	2.341	2.870
$A = 0.00125$	1.968	2.739	3.083

method of solution presented. The dynamic response of the rattling beam subject to an N-shaped pressure disturbance is studied in detail. This solution, aside from its intrinsic interest, should also provide some understanding of a loosely supported plate. It is believed that the effects of loose supports on plates subjected to sonic booms will be less severe than on beams since, after the first contact with the springs around the perimeter, new contacts or loss of contact will not occur simultaneously for all points of the perimeter. Thus, the effects of loose supports are likely to be diffused through the plate.

The maximum dynamic amplification factors (DAF) for moment and shear have been obtained for a continuous variation of R (from 1 to 10) and a given set of values for A, R_d, and K. They are graphically presented in Figs. 3 and 4. It is observed that for the cases considered, the DAF for moment exceeds 4, whereas the DAF for shear exceeds 6.65. Thus, both are much more severe than the ordinary case of simple beams constrained not to rattle. It is noteworthy that the shear is much more sensitive to an increase in spring constant (Table 1). This trend should continue for increasingly stiffer springs except that a point is probably quickly reached at which the Bernoulli-Euler beam theory becomes unreliable, and the more refined Timoshenko theory must be used for physically meaningful results.

Another source of difficulty is encountered in carrying out numerical computations. It stems from the fact that the time-variation of end deflection is significantly affected by higher frequencies for stiff springs. Consequently, the convergence for a five-term series used in all the computations becomes very slow, and it becomes increasingly more difficult to determine the times at which the beam leaves the spring or changes phases. The end deflection curve, as shown in Fig. 5, is for a relatively soft spring. It becomes much less smooth and crosses the base line (zero deflection) much more frequently and in an irregular manner when very stiff springs ($K = 0.0001$) are used. The same, of course, is true for the shear DAF in Fig. 6. It is interesting to note that the moment DAF shown in Fig. 6 is almost directly proportional to the difference of ordinates between the center and end deflections, as it should.

In Table 2, results due to a discrete variation of R_d within its range of interest is shown. It is noted that the center deflection, the DAF for shear and moment, changes only slightly for $R = 2.0$ and decreases when R_d is doubled for $R = 3.0$. In Table 3, the variation of DAF due to a discrete variation of the gap ratio is listed. It is interesting to note that although the DAF for both shear and moment increases with increase of A, the rate of increase is rather mild. Of course, the previous results are only valid for a given value of R. It would be of practical interest to study the DAF corresponding to different gap ratios for a continuous variation of R.

In conclusion, the effect of sonic boom on a beam that rattles is much more severe than on the one that does not. For the particular case studied, an additional factor of 2 is indicated for the DAF on the moment at midspan and about 4 on the shear at supports. This fact hints that a rattling beam would most likely fail near the supports instead of at the center, because the shear at supports which induces diagonal tension and compression stresses would become more critical than the bending stress induced by the moment at the center of the beam.

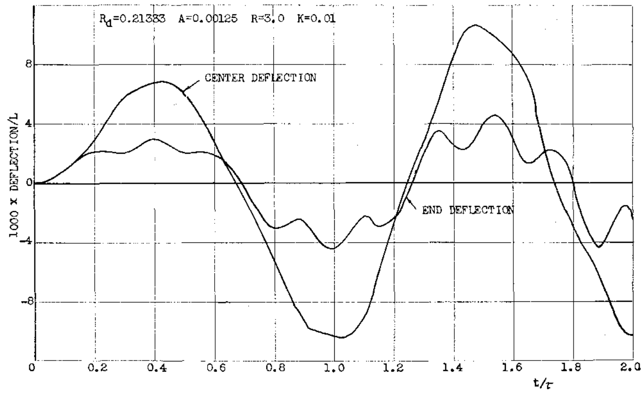

Fig. 5 Variation of deflections with time.

Appendix A: Frequencies and Normal Functions of a Spring-Supported Simple Beam

If the origin $x = 0$ is chosen at midspan of the beam, the frequencies and normal functions must satisfy the following equation:

$$EIZ'''' + \mu p^2 Z = 0 \tag{A1}$$

and the boundary conditions

$$Z_{(L/2)}'' = Z_{(-L/2)}'' = 0$$
$$EIZ_{(L/2)}''' = SZ_{(L/2)} \tag{A2}$$

or

$$EIZ_{(-L/2)}''' = -SZ_{(-L/2)}$$

Setting

$$L^4(\mu p^2/EI) = \beta^4 \tag{A3}$$

the solution of the differential equation is

$$Z = A\cosh(\beta x/L) + B\sinh(\beta x/L) + C\cos(\beta x/L) + D\sin(\beta x/L) \tag{A4}$$

Since only symmetric normal functions are of interest, we set $B = D = 0$. Substituting in the boundary conditions, one finds

$$A\cosh\beta/2 - C\cos\beta/2 = 0$$
$$EI\beta^3(A\sinh\beta/2 + C\sin\beta/2) = 3L^3(A\cosh\beta/2 + C\cos\beta/2) \tag{A5}$$

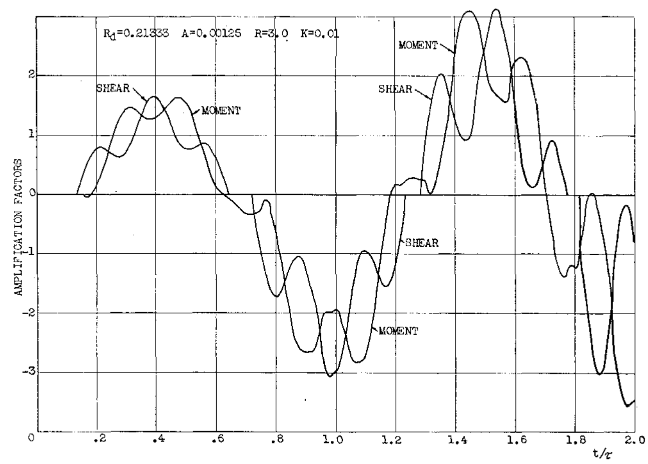

Fig. 6 Variation of M and Q with time.

Table 4 Frequencies of symmetric modes of spring-supported beams

	$p_n = (\beta_n^2/L^2)(EI/\mu)^{1/2}$				
K	β_1	β_2	β_3	β_4	β_5
∞[a]	0	4.73041	10.99561	17.27876	23.56194
10^4	0.118921	4.730043	10.99561	17.27876	23.56194
10^2	0.376045	4.730230	10.99562	17.27876	23.56195
1	1.184285	4.748888	10.99711	17.27915	23.56210
10^{-2}	2.876751	6.076171	11.15076	17.31802	23.57732
10^{-3}	3.111063	8.565595	12.57700	17.71093	23.72351
10^{-4}	3.138497	9.339894	15.28036	20.65647	25.53097
0	3.141593	9.424779	15.70797	21.99115	28.27434

[a] $K = \infty$ corresponds to free-free beam.

from (A5), the following frequency equation results:

$$\tanh\beta/2 + \tan\beta/2 = 2/K\beta^3 \tag{A6}$$

where

$$K = EI/SL^3 \tag{A7}$$

The first five roots of (A6) have been obtained for various values of K including those of a free-free beam for which K is infinitely large. The corresponding frequencies obtainable from (A3) are tabulated in Table 4.

The normal functions for the spring-supported beam can now be written as

$$Z = \cosh\frac{\beta x}{L}\Big/\cosh\frac{\beta}{2} + \cos\frac{\beta x}{L}\Big/\cos\frac{\beta}{2} \tag{A8}$$

Appendix B: Evaluation of Various Integrals Involving the Normal Functions

In the computation of the response of the beam, the following quantities are required: Eqs. (7), (9), and

$$c_n = \frac{1}{||Z_n||^2}\int_{-L/2}^{L/2} Z_n(x)dx \tag{B1}$$

The evaluation of (7) and (9) is straightforward. Using the expression (A8) and taking into account the frequency Eq. (A6) one finds

$$||Z||^2 = \frac{L}{2}\left[\frac{1}{\cosh^2(\beta_n/2)} + \frac{1}{\cos^2\beta_n/2} + \frac{12}{K\beta_n^4}\right] \tag{B2}$$

$$\beta_n \neq 0$$

$$T_{nm} = 8L/K(\beta_n^4 - \beta_m^4) \tag{B3}$$

In applying (B2) to the free-free mode, the value of K becomes infinite; hence, the last term in the bracket vanishes, provided β_n is different from zero. For the case $\beta_1 = 0$, $||Z_1||^2$ must be directly evaluated by noting from (A8) that $Z_1 = 2$; thus, $||Z_1||^2 = 4L$. In evaluating (B1), it is noted that for the free-free mode, (B1) may be written as

$$c_n = \frac{1}{2||Z_n||^2}\int_{-L/2}^{L/2} Z_n(x)Z_1dx \tag{B4}$$

which vanishes for all n except $n = 1$ because of the orthogonality properties of normal functions. It is obvious that for $n = 1$, $c_1 = \frac{1}{2}$.

For spring-supported modes, the values of c_n is evaluated by using (A8). Taking (A6) into account one finds

$$c_n = 4L/\beta_n^4K||Z_n||^2 \tag{B5}$$

References

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